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1988 J. Phys. A: Math. Gen. 21 L725

(http://iopscience.iop.org/0305-4470/21/14/003)

J. Phys. A: Math. Gen. 21 (1988) L725-L730. Printed in the UK

LETTER TO THE EDITOR

Canonical coherent-state representation of some squeeze operators

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Received 26 April 1988

Abstract. Using the IWOP technique (integration within an ordered product), we find canonical coherent-state representations of two kinds of squeeze operators. These representations manifestly show that they are quantum maps imaged by certain symplectic transformations in coordinate-momentum phase space. In terms of this formalism the coherent-state propagator of parametric amplifiers is easily obtained.

Squeezed states are now of considerable interest because of their potential application to precision interferometry, optical communication and gravitational wave detection [1]. In [2] a new approach for calculating the normally ordered form of the squeeze operators was introduced, which is based on the technique of 'integration within an ordered product' (IWOP) [3] and has the merit of showing the squeezing property from the outset. For the single-mode case, the squeeze operator has the following coherent-state representation:

$$S^{(1)}=\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\operatorname{sech} r)^{1/2}}|\mu p, q/\mu\rangle\langle p, q| \qquad \mu>0, \mu=\mathrm{e}^r \tag{1}$$

$$=\exp(-\tfrac{1}{2}a^{\dagger 2}\tanh r)\exp[-(a^\dagger a+\tfrac{1}{2})\ln\cosh r]\exp(\tfrac{1}{2}a^2\tanh r). \tag{1'}$$

Here $[a, a^\dagger]=1$ and

$$|z\rangle\equiv|p, q\rangle=\exp[\mathrm{i}(pQ-qP)]|0\rangle$$

$$=\exp[-\tfrac{1}{4}(q^2+p^2)+(1/\sqrt{2})(q+\mathrm{i}p)a^\dagger]|0\rangle \qquad z=\frac{q+\mathrm{i}p}{\sqrt{2}} \tag{2}$$

denotes the canonical coherent states [4], which satisfy

$$\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi}|p, q\rangle\langle p, q|=1.$$

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For the two-mode squeeze operator, the coherent-state representation is given by

$$S^{(2)}=\frac{\cosh\lambda}{(2\pi)^2}\int \mathrm{d}p_1\,\mathrm{d}q_1\,\mathrm{d}p_2\,\mathrm{d}q_2$$

$$\times\left|\left(\begin{matrix}\cosh\lambda & \sinh\lambda & 0 & 0\\ \sinh\lambda & \cosh\lambda & 0 & 0\\ 0 & 0 & \cosh\lambda & -\sinh\lambda\\ 0 & 0 & -\sinh\lambda & \cosh\lambda\end{matrix}\right)\left(\begin{matrix}q_1\\ q_2\\ p_1\\ p_2\end{matrix}\right)\right\rangle\left\langle\left(\begin{matrix}q_1\\ q_2\\ p_1\\ p_2\end{matrix}\right)\right| \tag{3}$$

$$=\exp(a^\dagger b^\dagger\tanh\lambda)\exp[(a^\dagger a+b^\dagger b+1)\ln\operatorname{sech}\lambda]\exp(-ab\tanh\lambda) \tag{3'}$$

where λ is real, and

$$\left|\left(\begin{matrix}q_1\\ q_2\\ p_1\\ p_2\end{matrix}\right)\right\rangle\equiv|p_1q_1,p_2q_2\rangle$$

$$=\exp[-\tfrac{1}{4}(p_1^2+q_1^2+p_2^2+q_2^2)+(a^\dagger/\sqrt{2})(q_1+\mathrm{i}p_1)$$

$$+(b^\dagger/\sqrt{2})(q_2+\mathrm{i}p_2)]|00\rangle. \tag{4}$$

In this letter, we give the canonical coherent-state representation for another kind of squeeze operator. By directly using the IWOP technique we want to show that the representation of $\exp[-\frac{1}{2}\mathrm{i}r(a^2+a^{\dagger 2})]\equiv U^{(1)}$ is given by

$$U^{(1)}=\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\operatorname{sech} r)^{1/2}}|p\cosh r-q\sinh r,\,q\cosh r-p\sinh r\rangle\langle p,q| \qquad r\text{ real} \tag{5}$$

while the representation of $\exp[\mathrm{i}\lambda(ab+a^\dagger b^\dagger)]\equiv U^{(2)}$ is given by

$$U^{(2)}=\frac{\cosh\lambda}{(2\pi)^2}\int \mathrm{d}p_1\,\mathrm{d}q_1\,\mathrm{d}p_2\,\mathrm{d}q_2$$

$$\times\left|\left(\begin{matrix}\cosh\lambda & 0 & 0 & -\sinh\lambda\\ 0 & \cosh\lambda & -\sinh\lambda & 0\\ 0 & -\sinh\lambda & \cosh\lambda & 0\\ -\sinh\lambda & 0 & 0 & \cosh\lambda\end{matrix}\right)\left(\begin{matrix}q_1\\ q_2\\ p_1\\ p_2\end{matrix}\right)\right\rangle\left\langle\left(\begin{matrix}q_1\\ q_2\\ p_1\\ p_2\end{matrix}\right)\right| \tag{6}$$

where the matrix denotes a symplectic transformation. The normally ordered form of $U^{(1)}$ and $U^{(2)}$ can thus be obtained by integrating (5) and (6). Since $U^{(1)}$ and $U^{(2)}$ are closely related to transformations generated by parametric amplifiers, this formalism leads to the coherent-state propagator for parametric amplifiers in a natural way.

Let us now consider the single-mode case.

Using (2) and

$$|0\rangle\langle 0|=:\exp(-a^\dagger a): \tag{7}$$

$$\exp(\lambda a^\dagger a)=:\exp[(\exp\lambda-1)a^\dagger a]: \tag{8}$$

as well as the IWOP technique, we have

$$U^{(1)}=\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\mathrm{sech}\,r)^{1/2}} \exp\{-\tfrac{1}{4}(p\cosh r-q\sinh r)^2-\tfrac{1}{4}(q\cosh r-p\sinh r)^2$$
$$+(a^{\dagger}/\sqrt{2})[(q\cosh r-p\sinh r)+\mathrm{i}(p\cosh r-q\sinh r)]\}|0\rangle$$
$$\times\langle 0|\exp\left(-\tfrac{1}{4}(q^2+p^2)+\frac{a}{\sqrt{2}}(q-\mathrm{i}p)\right)$$
$$=\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\mathrm{sech}\,r)^{1/2}}:\exp\left(-\tfrac{1}{2}(\cosh r)^2(q^2+p^2)+\frac{qp}{\sqrt{2}}\sinh 2r\right.$$
$$\left.+\frac{q}{\sqrt{2}}(a^{\dagger}\cosh r-\mathrm{i}a^{\dagger}\sinh r+a)+\frac{p}{\sqrt{2}}(\mathrm{i}a^{\dagger}\cosh r-a^{\dagger}\sinh r-\mathrm{i}a)-a^{\dagger}a\right):$$
$$=(\mathrm{sech}\,r)^{1/2}:\exp[-\tfrac{1}{2}\mathrm{i}\tanh r(a^2+a^{\dagger 2})+a^{\dagger}a(\mathrm{sech}\,r-1)]: \tag{9}$$
$$=\exp(-\tfrac{1}{2}\mathrm{i}a^{\dagger 2}\tanh r)\exp[(a^{\dagger}a+\tfrac{1}{2})\ln\mathrm{sech}\,r]\exp(-\tfrac{1}{2}\mathrm{i}a^2\tanh r). \tag{9'}$$

The expression for $U^{(1)}$ in (5) can also be expressed with the help of the symplectic transformation R defined by

$$\begin{pmatrix}q\\p\end{pmatrix}\to\begin{pmatrix}\cosh r & -\sinh r\\ -\sinh r & \cosh r\end{pmatrix}\begin{pmatrix}q\\p\end{pmatrix}\equiv R\begin{pmatrix}q\\p\end{pmatrix}$$
$$|p\cosh r-q\sinh r,\,q\cosh r-p\sinh r\rangle=\left|R\begin{pmatrix}q\\p\end{pmatrix}\right\rangle$$
$$U^{(1)}=\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\mathrm{sech}\,r)^{1/2}}\left|R\begin{pmatrix}q\\p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\p\end{pmatrix}\right|. \tag{10}$$

Since $\det R=1$, we have

$$U^{(1)\dagger}(R)=\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\mathrm{sech}\,r)^{1/2}}\left|\begin{pmatrix}q\\p\end{pmatrix}\right\rangle\left\langle R\begin{pmatrix}q\\p\end{pmatrix}\right|$$
$$=\int\frac{\mathrm{d}p\,\mathrm{d}q}{2\pi(\mathrm{sech}\,r)^{1/2}}\left|R^{-1}\begin{pmatrix}q\\p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}q\\p\end{pmatrix}\right|=U^{(1)}(R^{-1}). \tag{11}$$

Differentiating (9) with respect to r and using the following relations:

$$\exp(\nu a^{\dagger 2})a=(a-2\nu a^{\dagger})\exp(\nu a^{\dagger 2}) \tag{12}$$
$$\exp(\nu a^{\dagger 2})a^2=(a^2+4\nu^2a^{\dagger 2}-4\nu a^{\dagger}a-2\nu)\exp(\nu a^{\dagger 2}) \tag{13}$$

we obtain

$$\frac{\partial U^{(1)}}{\partial r}=-\tfrac{1}{2}\mathrm{i}a^{\dagger 2}(\mathrm{sech}\,r)^2U^{(1)}-\tanh r\exp(-\tfrac{1}{2}\mathrm{i}a^{\dagger 2}\tanh r)(a^{\dagger}a+\tfrac{1}{2})$$
$$\times\exp[(a^{\dagger}a+\tfrac{1}{2})\ln\mathrm{sech}\,r]\exp(-\tfrac{1}{2}\mathrm{i}a^2\tanh r)-\tfrac{1}{2}\mathrm{i}U^{(1)}a^2(\mathrm{sech}\,r)^2$$
$$=-\tfrac{1}{2}\mathrm{i}(a^2+a^{\dagger 2})U^{(1)}. \tag{14}$$

With the boundary condition $U^{(1)}(r=0)=1$ and (11), we find that

$$U^{(1)}=\exp[-\tfrac{1}{2}\mathrm{i}r(a^2+a^{\dagger 2})] \tag{15}$$
$$U^{(1)\dagger}(R)=U^{(1)^{-1}}(R)=U^{(1)}(R^{-1}) \tag{15'}$$

which indicates that $U^{(1)}$ is unitary. Instead, when r is time dependent, on the other hand, following the same procedures as (12)-(15) we learn that

$$\frac{\partial U^{(1)}(t,t_0)}{\partial t}=-\tfrac{1}{2}\mathrm{i}(a^2+a^{\dagger 2})U^{(1)}(t,t_0)\frac{\partial r}{\partial t} \qquad U^{(1)}(t_0,t_0)=1 \qquad r(t_0)=0. \tag{16}$$

If $\partial r/\partial t\equiv 2f(t)$, it follows that

$$\begin{aligned}\mathrm{i}\frac{\partial U^{(1)}(t,t_0)}{\partial t}&=f(t)(a^2+a^{\dagger 2})U^{(1)}(t,t_0)\\&\equiv f(t)\exp(\mathrm{i}H_0t)[a^{\dagger 2}\exp(-2\mathrm{i}\omega t)+a^2\exp(2\mathrm{i}\omega t)]\exp(-\mathrm{i}H_0t)U^{(1)}(t,t_0)\\&=H_1(t)U^{(1)}(t,t_0)\end{aligned} \tag{17}$$

where

$$H_1(t)\equiv\exp(\mathrm{i}H_0t)f(t)[a^{\dagger 2}\exp(-2\mathrm{i}\omega t)+a^2\exp(2\mathrm{i}\omega t)]\exp(-\mathrm{i}H_0t) \tag{18}$$

$$H_0\equiv\omega a^\dagger a. \tag{19}$$

From (15′)-(19) we can identify $U^{(1)}(t,t_0)$ with a unitary time evolution operator in the interaction picture. Thus, in the Schrödinger picture the Hamiltonian which generates squeezing is given by

$$H_{\mathrm{S}}=\omega a^\dagger a+f(t)[a^{\dagger 2}\exp(-2\mathrm{i}\omega t)+a^2\exp(2\mathrm{i}\omega t)]. \tag{20}$$

When $f(t)=K$, a constant, (20) is just the standard Hamiltonian of a degenerate parametric amplifier discussed, for example, in [5]. According to a standard transformation, the time evolution operator in the Schrödinger picture is given by

$$U_{\mathrm{S}}^{(1)}(t,t_0)=\exp(-\mathrm{i}H_0t)U^{(1)}(t,t_0)\exp(\mathrm{i}H_0t_0) \tag{21}$$

and specifically, for (20) we have

$$\begin{aligned}U_{\mathrm{S}}^{(1)}(t,t_0)&=\exp[-\tfrac{1}{2}\mathrm{i}a^{\dagger 2}\exp(-2\mathrm{i}\omega t)\tanh r]\exp\{a^\dagger a[\mathrm{i}\omega(t_0-t)+\ln\operatorname{sech} r]\}\\&\quad\times\exp[-\tfrac{1}{2}\mathrm{i}a^2\exp(2\mathrm{i}\omega t_0)\tanh r](\operatorname{sech} r)^{1/2}\\ r&\equiv 2\int_{t_0}^{t}f(t')\,\mathrm{d}t'.\end{aligned} \tag{22}$$

When $f(t)=K$, we are led to the coherent-state representation of the propagator for a degenerate parametric amplifier

$$\begin{aligned}\langle z,t|z_0,t_0\rangle&\equiv\langle z|U_{\mathrm{S}}(t,t_0)|z_0\rangle\\&=\exp\{-\tfrac{1}{2}\mathrm{i}\tanh[2K(t-t_0)][z^{*2}\exp(-2\mathrm{i}\omega t)+z_0^2\exp(2\mathrm{i}\omega t_0)]\\&\quad-\tfrac{1}{2}(|z|^2+|z_0|^2)+z^*z_0\exp[-\mathrm{i}\omega(t-t_0)]\\&\quad\times\operatorname{sech}[2K(t-t_0)]\}\operatorname{sech}\{[2K(t-t_0)]^{1/2}\}.\end{aligned} \tag{23}$$

In [5], this result was obtained by path integral techniques.

Now we consider the two-mode case.

The normally ordered form of $U^{(2)}$ in (6) can be obtained by using the IWOP technique and

$$|00\rangle\langle 00|=:\exp(-a^\dagger a-b^\dagger b):$$

namely

$$U^{(2)} = \cosh\lambda \int \frac{dp_1\, dp_1\, dp_2\, dq_2}{4\pi^2} :\exp\{-\tfrac{1}{4}(p_1'^2 + q_1'^2 + p_2'^2 + q_2'^2 + p_1^2 + q_1^2 + p_2^2 + q_2^2)$$

$$+ (1/\sqrt{2})[(q_1' + ip_1')a^\dagger + (q_2' + ip_2')b^\dagger + (q_1 - ip_1)a$$

$$+ (q_2 - ip_2)b] - a^\dagger a - b^\dagger b\}:$$

$$= \operatorname{sech}\lambda :\exp(-ia^\dagger b^\dagger \tanh\lambda)\exp[(\operatorname{sech}\lambda - 1)(a^\dagger a + b^\dagger b)]$$

$$\times \exp(-iab\tanh\lambda):$$

$$= \exp(-ia^\dagger b^\dagger \tanh\lambda)\exp[(a^\dagger a + b^\dagger b + 1)\ln\operatorname{sech}\lambda]\exp(-iab\tanh\lambda) \qquad (24)$$

where

$$p_1' = -q_2\sinh\lambda + p_1\cosh\lambda \qquad p_2' = p_2\cosh\lambda - q_1\sinh\lambda$$
$$q_1' = q_1\cosh\lambda - p_2\sinh\lambda \qquad q_2' = q_2\cosh\lambda - p_1\sinh\lambda. \qquad (25)$$

Following the same procedure as used in deriving (14), we find that

$$\partial U^{(2)}/\partial\lambda = -i(a^\dagger b^\dagger + ab)U^{(2)} \qquad U^{(2)}(\lambda = 0) = 1 \qquad (26)$$

which implies that

$$U^{(2)} = \exp[-i(a^\dagger b^\dagger + ab)\lambda] \qquad (27)$$

$$U^{(2)\dagger} = U^{(2)^{-1}}. \qquad (27')$$

Instead, when λ is time dependent, we have

$$\frac{\partial U^{(2)}(t, t_0)}{\partial t} = -i(a^\dagger b^\dagger + ab)U^{(2)}(t, t_0)\frac{\partial\lambda}{\partial t} \qquad U^{(2)}(t_0, t_0) = 1 \qquad \lambda(t_0) = 0. \qquad (28)$$

Let $\partial\lambda/\partial t = g(t)$, then (28) becomes

$$i\partial U^{(2)}(t, t_0)/\partial t = g(t)(ab + a^\dagger b^\dagger)U^{(2)}(t, t_0)$$

$$= g(t)\exp(i\mathbb{H}_0 t)[ab\exp(i\omega_s t) + a^\dagger b^\dagger \exp(-i\omega_s t)]\exp(-i\mathbb{H}_0 t)U^{(2)}(t, t_0)$$

$$= \mathbb{H}_I(t)U^{(2)}(t, t_0) \qquad (29)$$

where

$$\mathbb{H}_I(t) \equiv \exp(i\mathbb{H}_0 t)g(t)[ab\exp(i\omega_3 t) + a^\dagger b^\dagger \exp(-i\omega_3 t)]\exp(-i\mathbb{H}_0 t) \qquad (30)$$

$$\mathbb{H}_0 \equiv \omega_1 a^\dagger a + \omega_2 b^\dagger b \qquad \omega_3 \equiv \omega_1 + \omega_2. \qquad (31)$$

Similar to (17), we find the Hamiltonian of the parametric amplifier in the Schrödinger picture

$$\mathbb{H}_S = \omega_1 a^\dagger a + \omega_2 b^\dagger b + g(t)[ab\exp(i\omega_3 t) + a^\dagger b^\dagger \exp(-i\omega_3 t)]. \qquad (32)$$

The evolution operator in the Schrödinger picture is given by

$$U_S^{(2)}(t, t_0) = \exp(-i\mathbb{H}_0 t)U^{(2)}(t, t_0)\exp(i\mathbb{H}_0 t_0)$$

$$= \operatorname{sech}\lambda\,\exp[-ia^\dagger b^\dagger \exp(-i\omega_3 t)\tanh\lambda]$$

$$\times \exp[(a^\dagger a + b^\dagger b)\ln\operatorname{sech}\lambda + i(t_0 - t)\mathbb{H}_0]\exp[-iab\exp(i\omega_3 t_0)\tanh\lambda] \qquad (33)$$

from which the coherent-state representation of the propagator for the parametric amplifier is directly obtained, for $g(t) = K$, as

$$\begin{aligned}\langle z_1', z_2', t | z_1, z_2, t_0\rangle &\equiv \langle z_1', z_2' | U_S^{(2)}(t, t_0) | z_1, z_2\rangle \\ &= \operatorname{sech}[K(t-t_0)] \exp(-\tfrac{1}{2}(|z_1|^2 + |z_2|^2 + |z_1'|^2 + |z_2'|^2) \\ &\quad - \mathrm{i} \tanh[K(t-t_0)][z_2'^* z_1'^* \exp(-\mathrm{i}\omega_3 t) + z_1 z_2 \exp(\mathrm{i}\omega_3 t_0)] \\ &\quad - \operatorname{sech}[K(t-t_0)]\{\exp[\mathrm{i}\omega_1(t_0 - t)] z_1'^* z_1 + \exp[\mathrm{i}\omega_2(t_0 - t)] z_2'^* z_2\})\end{aligned} \tag{34}$$

which agrees with (48) in [5] (except that they have a factor of $\frac{1}{2}$ in the terms which include $\tanh[K(t-t_0)]$).

In summary, we see that the canonical coherent states and the IWOP technique provide a convenient way to study squeeze operators that are generated by ideal parametric amplifiers.

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